Shutterstock.com/MarjanCermelj

6

COMPLEX NUMBERS

COMPLEX ROOTS AND POLYNOMIALS

TERMINOLOGY

Argand diagram
argument
complex conjugate
complex polynomial
cubic
De Moivre's theorem
degree
discriminant
dividend
division identity
divisor
factor theorem
leading term
modulus
modulus-argument form
principal argument
quadratic
quartic
quintic
quotient
remainder
remainder theorem
roots of a complex number
roots of unity

Prior learning

ROOTS OF COMPLEX NUMBERS

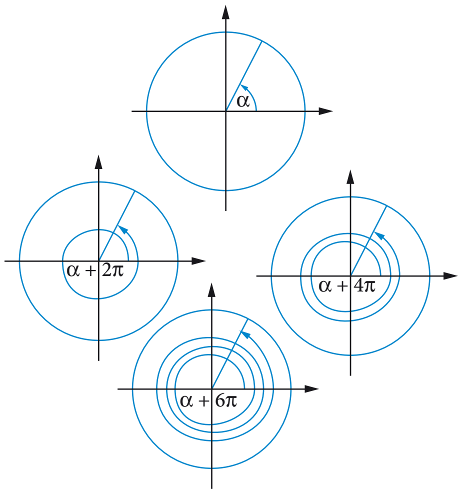
- determine and examine the n^{th} roots of unity and their location on the unit circle (ACMSM087)
- determine and examine the n^{th} roots of complex numbers and their location in the complex plane (ACMSM088)

FACTORISATION OF POLYNOMIALS

- prove and apply the factor theorem and the remainder theorem for polynomials (ACMSM089)
- consider conjugate roots for polynomials with real coefficients (ACMSM090)
- solve simple polynomial equations (ACMSM091)

6.01 ROOTS OF UNITY

The square root of 16 means the positive square root of 16, so $\sqrt{16} = 4$. However, there are two solutions of the equation $x^2 = 16$, $x = 4$ or $x = -4$.



In the complex number system there are two solutions to $z^2 = 1$: $z = 1$ or $z = -1$.

You can use **De Moivre's theorem** to see this more clearly.

First look at co-terminal angles for a complex number z.

Write z in **modulus-argument form** as $r[\cos(\alpha) + i\sin(\alpha)]$.

You can add any multiple of 2π to the **argument** without changing the number, so you get:

$$r[\cos(\alpha) + i\sin(\alpha)] = r[\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi)] \text{ for any } k \in \boldsymbol{Z}.$$

Then, to solve $z^2 = 1$, write $z^2 = 1[\cos(0 + 2k\pi) + i\sin(0 + 2k\pi)]$.

Let the solution be $z = r[\cos(\theta) + i\sin(\theta)]$.

Then using De Moivre's theorem, $z^2 = r^2[\cos(2\theta) + i\sin(2\theta)]$, so the equation $z^2 = 1$ becomes

$$r^2[\cos(2\theta) + i\sin(2\theta)] = 1[(\cos(2k\pi) + i\sin(2k\pi)] \text{ for any } k \in \boldsymbol{Z}.$$

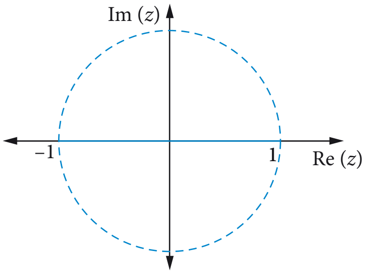
This gives $r^2 = 1$ and $2\theta = 2k\pi$, so $r = 1$ and $\theta = k\pi$ for any $k \in \boldsymbol{Z}$ and the possible solutions are: …, $1 \operatorname{cis}(-2\pi)$, $1 \operatorname{cis}(-1\pi)$, $1 \operatorname{cis}(0\pi)$, $1 \operatorname{cis}(1\pi)$, $1 \operatorname{cis}(2\pi)$, $1 \operatorname{cis}(3\pi)$, …

The answers repeat every 2π, so you should choose the answers whose argument is the **principal argument** in the domain $(-\pi, \pi)$.

For this equation: $1 \operatorname{cis}(0\pi) = 1$ or $1 \operatorname{cis}(1\pi) = -1$.

9780170250306

On an **Argand diagram**, these are opposite radii of the unit circle.



You can use the same procedure for higher **roots of unity** (1).

Example 1

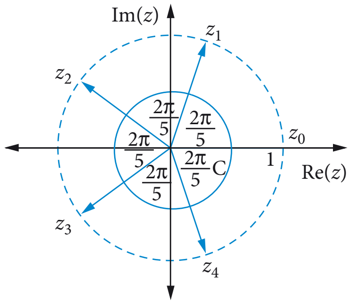
a Solve $z^5 = 1$.

b Show the roots on an Argand diagram and comment on their positions.

Solution

a Write z in modulus-argument form.	Let $z = r[\cos(\theta) + i\sin(\theta)]$
Write the equation.	$\{r[\cos(\theta) + i\sin(\theta)]\}^5 = 1 \text{ cis } (0)$
Write with all possible arguments.	$\{r[\cos(\theta) + i\sin(\theta)]\}^5 = 1 \text{ cis } (0 + 2k\pi)$ for $k \in \mathbf{Z}$.
Use De Moivre's theorem.	$r^5[\cos(5\theta) + i\sin(5\theta)]$ $= 1[\cos(2k\pi) + i\sin(2k\pi)]$
Isolate the modulus and argument.	$r^5 = 1$ and $5\theta = 2k\pi$ for $k \in \mathbf{Z}$
Solve to find them.	$r = 1$ and $\theta = \dfrac{2k\pi}{5}$ for $k \in \mathbf{Z}$
Write the principal values.	$r = 1$ and $\theta = \dfrac{2k\pi}{5}$ for $k = -2, -1, 0, 1, 2$.
Write out the solutions.	$z = \cos\left(-\dfrac{4\pi}{5}\right) + i\sin\left(-\dfrac{4\pi}{5}\right)$ or $z = \cos\left(-\dfrac{2\pi}{5}\right) + i\sin\left(-\dfrac{2\pi}{5}\right)$ or $z = \cos(0) + i\sin(0) = 1$ or $z = \cos\left(\dfrac{2\pi}{5}\right) + i\sin\left(\dfrac{2\pi}{5}\right)$ or $z = \cos\left(\dfrac{4\pi}{5}\right) + i\sin\left(\dfrac{4\pi}{5}\right)$

b Show the solutions on an Argand diagram.



The roots are shown as z_0, z_1, z_2, z_3, z_4.

Comment on the positions.

The roots are equally spaced around the unit circle, separated by angles of $\frac{2\pi}{5}$, with one root being 1.

From the Argand diagram of the 5th roots of 1, you can see that z_1 and z_4 are **complex conjugates** because their real parts are the same but their complex parts are opposite in sign. Since both have magnitude 1, this also means that they are inverses, so $z_1 z_4 = 1$. In fact, for any n, the nth roots of unity must be 1, −1 or multiply with another root to equal 1.

Example 2

Show that the 5th roots of unity are either 1 or have a product with another root to equal 1.

Solution

Write the 5th roots of 1.

$z_0 = \cos(0) + i\sin(0) = 1$ or

$z_1 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$ or

$z_2 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$ or

$z_3 = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$ or

$z_4 = \cos\left(-\frac{2\pi}{5}\right) \times i\sin\left(-\frac{2\pi}{5}\right)$

Write the result for z_0.

$z_0 = 1$

Write out $z_1 z_4$.

$z_1 z_4 = \left[\text{cis}\left(\frac{2\pi}{5}\right) \times \text{cis}\left(-\frac{2\pi}{5}\right)\right]$

$= \text{cis}(0)$

Multiply by adding the arguments.

$= \cos(0) + i\sin(0)$

$= 1$

Write out $z_2 z_3$.

$z_2 z_3 = \left[\text{cis}\left(\frac{4\pi}{5}\right) \times \text{cis}\left(-\frac{4\pi}{5}\right)\right]$

$= \text{cis}(0)$

Multiply by adding the arguments.

$= \cos(0) + i\sin(0)$

$= 1$

Write the overall result.

$z_0 = 1$, $z_1 z_4 = 1$ and $z_2 z_3 = 1$, so the 5th roots of unity are either 1 or multiply with another root to equal 1.

9780170250306

You can see from the first two examples that the roots of 1 have a particular pattern.

IMPORTANT

The nth **roots of unity** are complex numbers z such that $z^n = 1$. They lie on the unit circle and have the form $\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$, where $k = 0, 1, 2, \ldots, n-1$, with principal values $\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$, where $-\pi < \frac{2k\pi}{n} \le \pi$.

You can use a CAS calculator to calculate the roots of 1, but the format of the answers can be unsimplified. It is recommended that if you use the TI-Nspire for complex numbers, you use the rectangular form of complex numbers. The ClassPad assumes this form anyway. To express an answer in the form $r[\cos(\theta) + i\sin(\theta)]$, you can find the magnitude and argument of the answer(s). The ClassPad has a conversion to this form, which is called the trigonometric form on the calculator.

Example 3

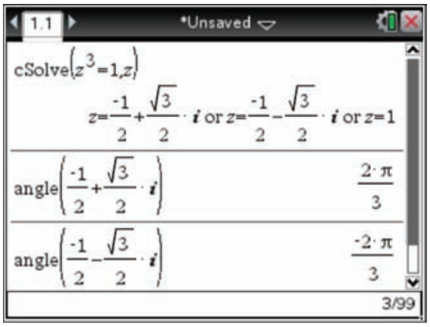
CAS Find the cube roots of 1.

Solution

TI-Nspire CAS

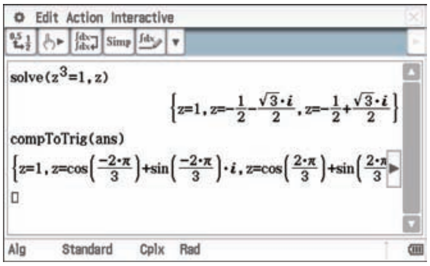
Change the document settings for Real or Complex to rectangular. Use menu, 3: Algebra, C: Complex and 1: Solve or type csolve ($z^3 = 1, z$).

Find the argument using menu, 2: Number, 9: Complex Number Tools, 4: Polar Angle or type angle() and copy and paste each answer into the brackets.



ClassPad

Ensure that the calculator is in **Cplex** (complex) mode rather than **Real**. Tap **Action**, **Advanced** and **solve** and then the equation and variable of interest, or type solve($z^3 = 1, z$). Change the answer to trigonometric form by tapping **Action**, **Complex**, **compToTrig**, then type ans) or type compToTrig(ans). Tap the right arrow ▶ to see the part of the solution that doesn't fit on the screen.



Write the answers, assuming that the magnitude is 1.

The cube roots are $\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$, 1 or $\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$.

If a question does not specify the form of an answer in complex numbers, you can choose to give the answer in either form, so choose the one that is simplest for that question.

INVESTIGATION Roots of unity

What do you get when you multiply or divide roots of unity?

- Find a cube root of unity and call it, say, z_3.
- Find an eighth root of unity and call it, say, z_8.
- Find the values of $z_3 \times z_8$ and $z_3 \div z_8$.
- Investigate powers of $z_3 \times z_8$ and $z_3 \div z_8$.
- What do you find?
- What suppositions can you make?
- Can you prove your suppositions?

EXERCISE 6.01 Roots of unity

Concepts and techniques

1 Example 1 a Solve $z^3 = 1$.
   b Show the roots on an Argand diagram and comment on their positions.
   c Express the roots in rectangular form.

2 a Solve $z^4 = 1$.
   b Show the roots on an Argand diagram and comment on their positions.

3 a Solve $z^6 = 1$.
   b Show the roots on an Argand diagram and comment on their positions.

4 Example 2 Show that the cube roots of unity are either 1 or multiply with another root to equal 1.

5 Show that the fourth roots of unity are 1, −1 or multiply with another root to equal 1.

6 Show that the 6th roots of unity are 1, −1 or when multiplied by another root are to equal 1.

7 Example 3 CAS a Find the fourth roots of 1 in rectangular form.
   b Change the roots to polar form.

8 CAS a Find the eighth roots of 1 in rectangular form.
   b Change the roots to polar form.

9 CAS a Find the sixth roots of 1 in rectangular form.
   b Change the roots to polar form.

Reasoning and communication

10 Show that if p is an nth root of 1, then p^m is also an nth root, where $n, m \in \mathbb{Z}^+$.

11 Show that if p and q are nth roots of 1, then pq is also an nth root, where $n \in \mathbb{Z}^+$.

9780170250306

12 Show that if p and q are nth roots of 1, then there exist positive integers a and b, where $a \neq b$, such that $p^a = q^b$.

13 For any positive rational number x, show that $\cos(x\pi) + i\sin(x\pi)$ is a power of an nth root of 1, for some $n \in \mathbf{Z}^+$.

14 Show that the nth roots of unity are 1, −1 or multiply by another root to equal 1.

6.02 ROOTS OF COMPLEX NUMBERS

Finding the **roots of complex numbers** is similar to finding the roots of unity. For numbers with a magnitude of 1, the only difference to the roots of unity is that you add $2k\pi$ to the argument not equal to 0.

Example 4

Find the 6th roots of i.

Solution

Write i in modulus-argument form.	$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$
Write z in modulus-argument form.	Let $z = r[\cos(\theta) + i\sin(\theta)]$
Write an equation.	$\{r[\cos(\theta) + i\sin(\theta)]\}^6 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$
Write with all possible arguments.	$\{r[\cos(\theta) + i\sin(\theta)]\}^6 = 1\text{ cis}\left(\frac{\pi}{2} + 2k\pi\right)$ for $k \in \mathbf{Z}$
Use De Moivre's theorem.	$r^6\text{ cis}(6\theta) = 1\text{ cis}\left(\frac{\pi}{2} + 2k\pi\right)$
Isolate the modulus and argument.	$r^6 = 1$ and $6\theta = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbf{Z}$
Solve to find them.	$r = 1$ and $\theta = \frac{\pi}{12} + \frac{2k\pi}{6}$ for $k \in \mathbf{Z}$
Write the principal arguments.	$r = 1$ and $\theta = \frac{\pi}{12} + \frac{k\pi}{3}$ for $k = -3, -2, -1, 0, 1, 2$
Write on a common denominator.	$r = 1$ and $\theta = \frac{(4k+1)\pi}{12}$ for $k = -3, -2, -1, 0, 1, 2$
Write out the solutions.	$z = \cos\left(\frac{-11\pi}{12}\right) + i\sin\left(\frac{-11\pi}{12}\right)$ or
	$z = \cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)$ or
	$z = \cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)$ or
	$z = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$ or
	$z = \cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)$ or
	$z = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

The answers to Example **4** could be given as

$$z = \cos\left[\frac{(4k+1)\pi}{12}\right] + i\sin\left[\frac{(4k+1)\pi}{12}\right] \text{ for } k = -3, -2, \ldots, 2.$$

You could also write it as

$$z = \cos\left[\frac{(4k+1)\pi}{12}\right] + i\sin\left[\frac{(4k+1)\pi}{12}\right] \text{ for } \{\, k \in \boldsymbol{Z}: -3 \le k \le 2\}$$

to avoid writing out each root separately.

Generally, when you find the roots of complex numbers, you have to take account of the magnitude as well. If the magnitude of the number is r, then the magnitude of the nth root will be $\sqrt[n]{r}$.

Example 5

a Find the cube roots of –125.

b Show the roots on the complex plane.

Solution

a Write –125 in modulus-argument form. $-125 = 125[\cos(\pi) + i\sin(\pi)]$

Write z in modulus-argument form. Let $z = r[\cos(\theta) + i\sin(\theta)]$

Write an equation. $[r\,\text{cis}\,(\theta)]^3 = 125\,\text{cis}\,(\pi)$

Write with all possible arguments. $[r\,\text{cis}\,(\theta)]^3 = 125\,\text{cis}\,(\pi + 2k\pi)$ for $k \in \boldsymbol{Z}$

Use De Moivre's theorem. $r^3\,\text{cis}\,(3\theta) = 125\,\text{cis}\,(\pi + 2k\pi)$

Isolate the modulus and argument. $r^3 = 125$ and $3\theta = \pi + 2k\pi$ for $k \in \boldsymbol{Z}$

Solve to find them. $r = 5$ and $\theta = \frac{\pi}{3} + \frac{2k\pi}{3}$ for $k \in \boldsymbol{Z}$

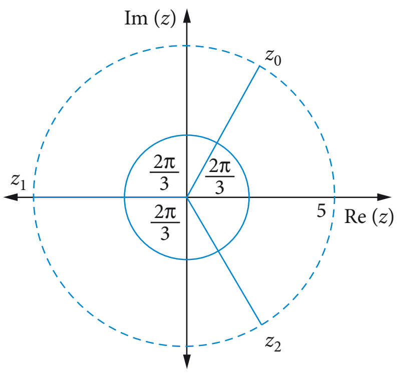
Write the principal values. $r = 5$ and $\theta = \frac{(2k+1)\pi}{3}$ for $k = -1, 0, 1$

Write out the solutions. $z = 5\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$ or

$z = 5\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ or

$z = 5[\cos(\pi) + i\sin(\pi)]$

b Show the solutions on an Argand diagram. They are all on a circle of radius 5, centred at the origin.



9780170250306

You will only get a rational or integer magnitude for a root when the magnitude of the original number happens to be the appropriate power.

Example 6

a Solve the equation $z^4 + i = 1$.

b Show the roots on the complex plane.

Solution

a Write in the form $z^n = w$.

$z^4 = 1 - i$

Write $1 - i$ in modulus argument form.

$1 - i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

Write z in modulus argument form.

Let $z = r[\cos(\theta) + i\sin(\theta)]$

Write an equation.

$[r \operatorname{cis}(\theta)]^4 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

Write with all possible arguments.

$[r \operatorname{cis}(\theta)]^4 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4} + 2k\pi\right)$ for $k \in \mathbf{Z}$

Use De Moivre's theorem.

$r^4 \operatorname{cis}(4\theta) = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4} + 2k\pi\right)$

Isolate the modulus and argument.

$r^4 = \sqrt{2}$ and $4\theta = -\frac{\pi}{4} + 2k\pi$ for $k \in \mathbf{Z}$

Solve to find them.

$r = \sqrt[8]{2}$ and $\theta = -\frac{\pi}{16} + \frac{2k\pi}{4}$ for $k \in \mathbf{Z}$

Write the principal values.

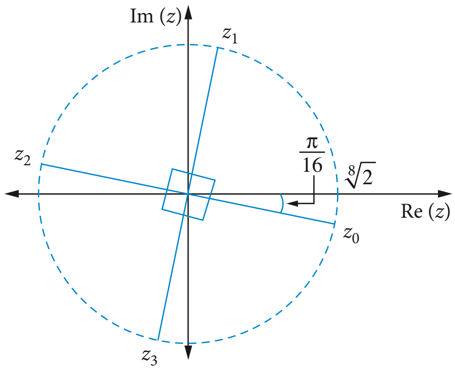
$r = \sqrt[8]{2}$ and $\theta = \frac{(8k-1)\pi}{16}$ for $k = -1, 0, 1, 2$

Write the solutions.

$z = \sqrt[8]{2}\left\{\cos\left[\frac{(8k-1)\pi}{16}\right] + i\sin\left[\frac{(8k-1)\pi}{16}\right]\right\}$

for $k = -1, 0, 1, 2$

b Show the solutions on an Argand diagram. They are all on a circle of radius $\sqrt[8]{2} \approx 1.0905$, centred at the origin.



You should write out the solutions in full for a small number of solutions, but for a large number you should use a variable such as k to give the solutions. Sometimes the way you express an answer can be different to someone else's expression, yet both will be correct, such as $\frac{(8k-1)\pi}{16}$ for $k = -1, 0, 1, 2$; $\frac{(8k+7)\pi}{16}$ for $k = -2, -1, 0, 1$; $k\pi - \frac{9\pi}{16}$, $k\pi - \frac{\pi}{16}$ for $k = 0, 1$; or $k\pi + \frac{7\pi}{16}$, $k\pi + \frac{11\pi}{16}$ for $k = -1, 0$. You should check that each of these expressions gives the same set of answers.

IMPORTANT

The nth **roots of a complex number** z are on a circle of radius $\sqrt[n]{|z|}$, separated from each other by the angle $\frac{2\pi}{n}$ and have the form

$$\sqrt[n]{|z|}\left\{\cos\left[\frac{\arg(z)+2k\pi}{n}\right]+i\sin\left[\frac{\arg(z)+2k\pi}{n}\right]\right\},\text{ where } k = 0, 1, 2, \ldots, n-1.$$

Once you have one root of a complex number, you can use the separation of the roots by the angle $\frac{2\pi}{n}$ to find the others. For example, if one 6th root of a complex number is $\sqrt{5}\text{ cis }(\frac{\pi}{4})$, then the other roots are separated by multiples of $\frac{\pi}{3}$ from this and each other, so they must be $\sqrt{5}\text{ cis }(-\frac{\pi}{12})$, $\sqrt{5}\text{ cis }(-\frac{5\pi}{12})$, $\sqrt{5}\text{ cis }(-\frac{3\pi}{4})$, $\sqrt{5}\text{ cis }(\frac{7\pi}{12})$ and $\sqrt{5}\text{ cis }(\frac{11\pi}{12})$.

EXERCISE 6.02 Roots of complex numbers

WS Roots of complex numbers

Concepts and techniques

1 Example 4 Find the 8th roots of $-i$.

2 Find the cube roots of $\frac{\sqrt{3}-i}{2}$.

3 Solve the equation $\sqrt{2}\, z^4 + 1 = i$.

4 Example 5 a Find the cube roots of $64i$.
  b Show the roots on the complex plane and comment on their positions.

5 a Find the fourth roots of -81.
  b Show the roots on an Argand diagram and comment on their positions.

6 a Solve the equation $z^2 + 7 = 24i$, with the argument correct to 4 decimal places.
  b Express in Cartesian form.
  c Show the solutions on the complex plane.

7 Example 6 a Find the exact sixth roots of $20 + 15i$.
  b Show the roots on the complex plane and comment on their positions.

8 a Find the fifth roots of $-64i$.
  b Show the roots on the complex plane and comment on their positions.

9 a Solve the equation $z^3 + 28 + 10i = 0$ and express the answers in Cartesian form correct to 3 decimal places.
  b Show the solutions on the complex plane and comment on their positions.

Reasoning and communication

10 Show that the $(2n)$th roots of any negative number are complex, where $n \in Z^+$. That is, that none of the roots are real or purely imaginary.

9780170250306

6.03 COMPLEX POLYNOMIALS

You already know a lot about polynomials with real coefficients. A **complex polynomial** is exactly the same, except that the coefficients can be complex numbers. The complex numbers include purely real and purely imaginary numbers.

IMPORTANT

A polynomial is a sum of terms containing non-negative integer powers of the variable.
A **complex polynomial** in z is of the form:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$$

where n is a positive whole number, $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ may be complex numbers and $a_n \neq 0$.

The highest power of z is called the **degree** of the polynomial. The term that contains the highest power is called the **leading term**.

The degree of the polynomial above is n, and the leading term is $a_n z^n$. You can find the value of a complex polynomial at particular points by substituting in exactly the same way as for real polynomials. Real polynomials form a subset of complex polynomials.

Example 7

The function $m(z) = 3z^3 - iz^2 + 2iz^4 - 5z - z^4 + 3iz + 7i$

a What is the degree of $m(z)$?

b What is the leading term?

Solution

a The highest power of z is z^4. $m(z)$ is of degree 4.

b Collect the coefficients in order. $m(z) = (-1 + 2i)z^4 + 3z^3 - iz^2 + (-5 + 3i)z + 7i$

Write the leading term. The leading term is $(-1 + 2i)z^4$.

Shutterstock.com/siambizkit

Example 8

Find the values of $p(z) = 3z^3 - iz^2 + 2i - 5$ at the points $z = 3$, $z = -2i$ and $z = 2 + 3i$.

Solution

Substitute $z = 3$.	$p(3) = 3 \times 3^3 - i \times 3^2 + 2i - 5$
Simplify.	$= 76 - 7i$
Substitute $z = -2i$.	$p(-2i) = 3 \times (-2i)^3 - i \times (-2i)^2 + 2i - 5$
Simplify each term.	$= 24i + 4i + 2i - 5$
Simplify and write the real part first.	$= -5 + 30i$
Substitute $z = 2 + 3i$.	$p(-2i) = 3 \times (2 + 3i)^3 - i \times (2 + 3i)^2 + 2i - 5$
Simplify each term.	$= 24 + 108i - 162 - 81i - 4i + 12 + 9i + 2i - 5$
Simplify and write the real part first.	$= -131 + 34i$

The arithmetic for complex polynomials is a bit more difficult than for real polynomials so you should check your calculations with a CAS calculator whenever possible.

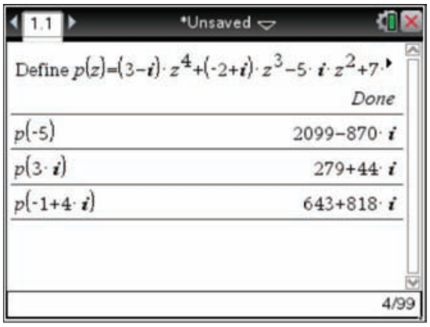
Example 9

CAS Given the function $p(z) = (3 - i)z^4 + (-2 + i)z^3 - 5iz^2 + 7z + 9 + 5i$, find $p(-5)$, $p(3i)$ and $p(-1 + 4i)$.

Solution

TI-Nspire CAS

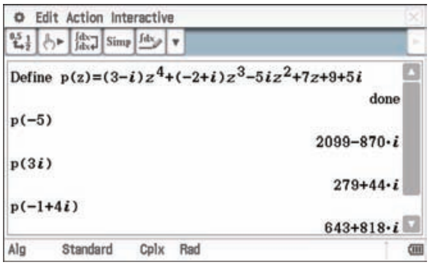
Set your calculator to rectangular (complex) mode and Define $p(z)$. Make sure that you use $\boldsymbol{i}$ from the symbols menu (π▸ or ctrl 📖) for the imaginary number i.
Then enter $p(-5)$, $p(3i)$ and $p(-1 + 4i)$.



9780170250306

ClassPad

Make sure that your calculator is set to **Cplx** and define $p(z)$. Press Keyboard and use the Math3 keyboard and Define $p(z)$. Make sure that you use the imaginary number i, (see i on the soft keyboards Math2 or Math3) not the letter i.
Then enter $p(-5)$, $p(3i)$ and finally $p(-1+4i)$.



Write the answers.

$p(-5) = 2099 - 870i$, $p(3i) = 279 + 44i$ and $p(-1+4i) = 643 + 818i$

EXERCISE 6.03 Complex polynomials

Complex polynomials

Concepts and techniques

1 Example 7 What is the degree of $p(z) = 3z^3 - iz^5 + 8iz^4 - 5z - z^3 + 7iz + 7i$?
A 3 B 4 C 5 D 7 E 8

2 What is the leading term of $p(z) = 6z^2 - 2iz^3 + 2iz^5 - 5z - iz^2 + 2iz^3 + 7i$?
A $6z^2$ B $(6-i)z^2$ C $(6-2i)z^5$ D $(2i-5)z^5$ E $2iz^5$

3 Example 8 Given the function $t(z) = 3z^3 - iz^2 + 2iz - 6 + 3i$, find each of the following.
a $t(2)$ b $t(2i)$ c $t(-3)$ d $t(-3i)$ e $t(2-3i)$

4 Given the function $m(z) = 3iz^4 + 5z^3 + (2i-3)z + 4i$, find each of the following.
a $m(-5)$ b $m(3i)$ c $m(-5+3i)$ d $m(4-i)$ e $m(2+i)$

5 Given the function $p(z) = (3-i)z^2 + (-2+5i)z + 4 - 2i$, find each of the following.
a $p(4)$ b $p(5i)$ c $p(4+5i)$ d $p(-5-2i)$ e $p(6+i)$

6 Example 9 CAS Given the function $q(z) = (5+3i)z^4 + (4-3i)z^2 - 8z - 5i$, find each of the following.
a $q(4)$ b $q(5i)$ c $q(4+5i)$ d $q(-5-2i)$ e $q(6+i)$

7 CAS Given the function $f(z) = (3-2i)z^3 + (5+6i)z^2 + (-4+5i)z - 4 + 8i$, find each of the following.
a $f(-3)$ b $f(6i)$ c $f(-3+6i)$ d $f(4-7i)$ e $f(2+4i)$

8 CAS Given the function $g(z) = (-2-4i)z^5 + (5+7i)z^3 - 8iz^2 + 3z - 4 + 6i$, find each of the following.
a $g(2+3i)$ b $g(-3+4i)$ c $g(3-2i)$ d $g(4+7i)$ e $g(-2-i)$

Reasoning and communication

9 In general, is it true that $p(a) + p(ib) = p(a+ib)$ for complex polynomials? Prove or disprove this proposition. Does it depend on the degree?

9780170250306

6.04 REMAINDER AND FACTOR THEOREMS

You can add, subtract, multiply and divide complex polynomials in the same way as real polynomials, but the coefficients are complex instead of real. When you divide one complex polynomial into another, remember to allow for both the real and imaginary part of the coefficient.

Example 10

Three functions are defined as follows:

$p(z) = (2 + i)z^3 - 5z^2 + 4 + 6i$,

$q(z) = (6 - 2i)z^2 + (-4 - 3i)z - 6$ and

$d(z) = z - 2 + 3i$.

a Find and simplify $p(z) - q(z)$.

b Find and simplify $p(z)q(z)$.

c Find and simplify $p(z) \div d(z)$.

Solution

a Write the difference.

$$p(z) - q(z)$$
$$= [(2 + i)z^3 - 5z^2 + 4 + 6i] - [(6 - 2i)z^2 + (-4 - 3i)z - 6]$$

Add the coefficients of each power.

$$= [(2 + i)z^3 + [-5 - (6 - 2i)]z^2 - (-4 - 3i)z + 4 + 6i - (-6)$$

Simplify.

$$= [(2 + i)z^3 + [-11 + 2i)]z^2 + (4 + 3i)z + 10 + 6i$$

b Write the product.

$$p(z)q(z)$$
$$= [(2 + i)z^3 - 5z^2 + 4 + 6i] \times [(6 - 2i)z^2 + (-4 - 3i)z - 6]$$

Use the distributive law.

$$= (2 + i)z^3 \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] - 5z^2 \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] + 4 \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] + 6i \times [(6 - 2i)z^2 + (-4 - 3i)z - 6]$$

Use the distributive law again.

$$= (12 - 4i + 6i + 2)z^5 + (-8 - 6i - 4i + 3)z^4 + (-12 - 6i)z^3 + (-30 + 10i)z^4 + (20 + 15i)z^3 + 30z^2 + (24 - 8i)z^2 + (-16 - 12i)z - 24 + (12 + 36i)z^2 + (18 - 24i)z - 36i$$

Collect like terms.

$$= (14 + 2i)z^5 - 35z^4 + (8 + 9i)z^3 + (66 + 28i)z^2 + (2 - 36i)z - 24 - 36i$$

9780170250306

c Write the quotient.

$$p(z) \div d(z) = \frac{(2+i)z^3 - 5z^2 + 4 + 6i}{z - 2 + 3i}$$

Set out as long division.

$$z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i}$$

Divide the z^3 term by z.

$$\begin{array}{r}
(2+i)z^2 \\
z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i}
\end{array}$$

Multiply.

$$\begin{array}{r}
(2+i)z^2 \\
z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
(2+i)z^3+(-7+4i)z^2
\end{array}$$

Subtract and bring down.

$$\begin{array}{r}
(2+i)z^2 \\
z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
\underline{(2+i)z^3+(-7+4i)z^2} \\
(2-4i)z^2+(0+0i)z+4+6i
\end{array}$$

Divide the z^2 term.

$$\begin{array}{r}
(2+i)z^2+(2-4i)z \\
z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
\underline{(2+i)z^3+(-7+4i)z^2} \\
(2-4i)z^2+(0+0i)z+4+6i
\end{array}$$

Multiply.

$$\begin{array}{r}
(2+i)z^2+(2-4i)z \\
z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
\underline{(2+i)z^3+(-7+4i)z^2} \\
(2-4i)z^2+(0+0i)z+4+6i \\
(2-4i)z^2+(8+14i)z
\end{array}$$

Subtract and bring down.

$$\begin{array}{r}
(2+i)z^2+(2-4i)z \\
z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
\underline{(2+i)z^3+(-7+4i)z^2} \\
(2-4i)z^2+(0+0i)z+4+6i \\
\underline{(2-4i)z^2+(8+14i)z} \\
-(8-14i)z+4+6i
\end{array}$$

Divide the z term.

$$\begin{array}{r}
(2+i)z^2+(2-4i)z \\
z-2+3i\overline{\big)(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
\underline{(2+i)z^3+(-7+4i)z^2} \\
(2-4i)z^2+(0+0i)z+4+6i \\
\underline{(2-4i)z^2+(8+14i)z} \\
-(8-14i)z+4+6i
\end{array}$$

Multiply.

$$
\begin{array}{r}
(2+i)z^2+(2-4i)z-8-14i \\
z-2+3i\,\overline{)\,(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
\underline{(2+i)z^3+(-7+4i)z^2} \\
(2-4i)z^2+(0+0i)z+4+6i \\
\underline{(2-4i)z^2+(8+14i)z} \\
-(8-14i)z+4+6i \\
-(8-14i)z+58+4i
\end{array}
$$

Subtract. There is no more to bring down.

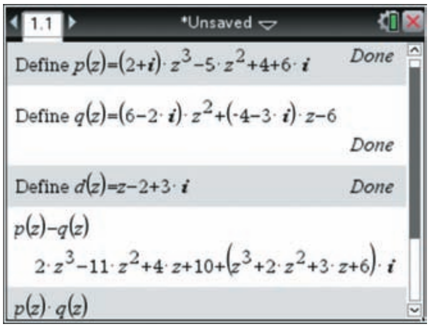
$$
\begin{array}{r}
(2+i)z^2+(2-4i)z-8-14i \\
z-2+3i\,\overline{)\,(2+i)z^3+(-5+0i)z^2+(0+0i)z+4+6i} \\
\underline{(2+i)z^3+(-7+4i)z^2} \\
(2-4i)z^2+(0+0i)z+4+6i \\
\underline{(2-4i)z^2+(8+14i)z} \\
-(8+14i)z+4+6i \\
\underline{-(8+14i)z+58+4i} \\
-54+2i
\end{array}
$$

Write in standard form.

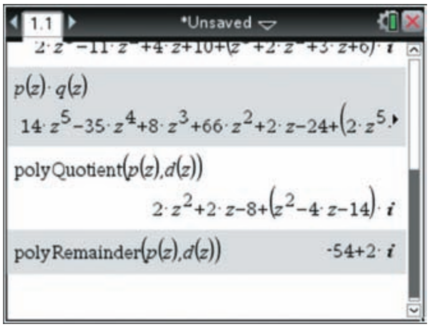
$$(p \div d)(z) = (2+i)z^2 + (2-4i)z - 8 - 14i + \frac{-54+2i}{z-2+3i}$$

TI-Nspire CAS

Set to complex (rectangular) mode and define the functions.
You can add, subtract and multiply them.



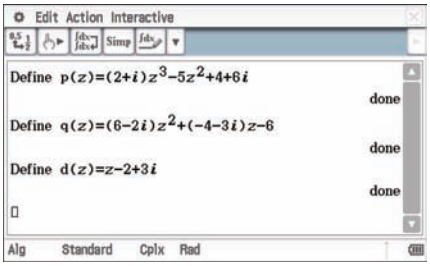
Use [menu], 3: Algebra, 8: Polynomial Tools and 5: Quotient of Polynomial to find the quotient.
Use 4: Remainder of polynomial to get the remainder.



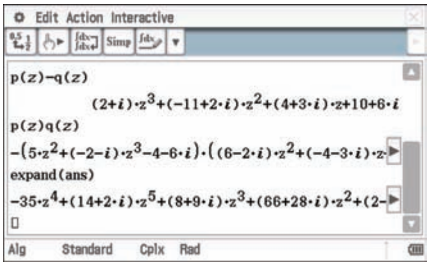
9780170250306

ClassPad

In **Cplx** mode, define the functions. You can add, subtract and multiply them.



Perform the subtraction and multiplication operations. Tap **Action**, **Transformation**, then **expand** to see the product in standard form.



The ClassPad does not have polynomial tools.

In Example **10**, the difference of the polynomials has the higher degree of the two. The **remainder** was written in standard form as a fraction with the denominator of $z - 2 + 3i$. This is similar to writing $13 \div 5 = 2\frac{3}{5}$ or $2504 \div 23 = 108\frac{20}{23}$. The relationship between the **quotient**, **divisor**, **dividend** and remainder is the same.

WS

Polynomial operations

A difference or sum of polynomials could give a result of lower degree than either polynomial if they were of the same degree and had leading terms with the same coefficient or the same but for the sign.

IMPORTANT

If a complex polynomial $P(z)$ is divided by the complex polynomial $D(z)$, then the **remainder** $R(z)$ has degree less than $D(z)$. The **quotient** $Q(z)$, **divisor**, remainder and **dividend** are related by the **division identity** $P(z) = D(z)Q(z) + R(z)$.

For a divisor of degree one, the remainder theorem for complex polynomials is the same as that for real polynomials.

IMPORTANT

Remainder theorem

If a complex polynomial $P(z)$ is divided by $D(z) = z - a$, where $a \in \mathbf{C}$, then the remainder is given by $R = P(a)$.

From the division identity, $P(z) = (z - a)Q(z) + R(z)$.

Since $R(z)$ has degree less than that of $(z - a)$, it is of degree 0, so it is a constant, say R.

Substituting $z = a$ gives $P(a) = (a - a)Q(a) + R$, which gives the required result. **QED**

Example 11

The function $g(z) = (6 + 2i)z^4 + (-2 + 3i)z^2 + (4 - i)z - 4 - 5i$. Find the remainders when $g(z)$ is divided by:

a $z + 3$ **b** $z - 2i$ **c** $z - 4 + 2i$

Solution

a Use the remainder theorem.

$$g(-3) = (6 + 2i)(-3)^4 + (-2 + 3i)(-3)^2 + (4 - i) \times (-3) - 4 - 5i$$

Simplify.

$$= 486 + 162i - 18 + 27i - 12 + 3i - 4 - 5i$$

Write the answer.

Remainder $= 452 + 187i$

b Use the remainder theorem.

$$g(2i) = (6 + 2i)(2i)^4 + (-2 + 3i)(2i)^2 + (4 - i) \times (2i) - 4 - 5i$$

Simplify.

$$= 96 + 32i + 8 - 12i + 8i + 2 - 4 - 5i$$

Write the answer.

Remainder $= 102 + 23i$

c Use the remainder theorem.

$$g(4 - 2i) = (6 + 2i)(4 - 2i)^4 + (-2 + 3i)(4 - 2i)^2 + (4 - i) \times (4 - 2i) - 4 - 5i$$

Simplify.

$$= (6 + 2i)(-112 - 384i) + (-2 + 3i)(12 - 16i) + 16 - 12i - 2 - 4 - 5i$$

Keep going.

$$= 96 - 2528i + 24 + 68i + 10 - 17i$$

Write the answer.

$$= 130 - 2477i$$

The factor theorem follows logically from the remainder theorem.

IMPORTANT

Factor theorem

The expression $z - a$, where $a \in \boldsymbol{C}$, is a factor of the complex polynomial $P(z)$ if and only if $P(a) = 0$.

From the remainder theorem, the remainder when $P(z)$ is divided by $z - a$ is $R = P(a)$.

If $z - a$ is a factor, then the remainder is 0, so $P(a) = 0$.

If $P(a) = 0$, then the remainder is zero, so $z - a$ is a factor. **QED**

You can use the factor theorem to show that a linear expression is a factor of a complex polynomial in the same way as for real polynomials.

9780170250306

Example 12

Show that $z + 4 - i$ is a factor of $(2 + 3i)z^3 + (11 + 11i)z^2 + (-2 - i)z - 17 - 17i$.

Solution

Write the polynomial.	$P(z) = (2 + 3i)z^3 + (11 + 11i)z^2 + (-2 - i)z - 17 - 17i$
Find $P(-4 + i)$	$P(-4 + i) = (2 + 3i)(-4 + i)^3 + (11 + 11i)(-4 + i)^2 + (-2 - i)(-4 + i) - 17 - 17i$
Simplify.	$= (2 + 3i)(-64 + 48i + 12 - i) + (11 + 11i)(16 - 8i - 1) + (8 - 2i + 4i + 1) - 17 - 17i$
Keep going.	$= -245 - 62i + 253 + 77i - 8 - 15i$
Still going!	$= 0$
Write the result.	$P(-4 + i) = 0$, so by the factor theorem, $z + 4 - i$ is a factor of $P(z)$.

EXERCISE 6.04 Remainder and factor theorems

Concepts and techniques

WS Remainder and factor theorems

1 Example 10 Three functions are defined as follows:
$p(z) = (-3 - 2i)z^3 + (5 + 2i)z^2 + (1 - 2i)z + 8 + 3i$,
$q(z) = (4 + 5i)z^2 + (3 - 2i)z + 4 - 6i$ and
$g(z) = (2 + 2i)z^2 + (-2 - 4i)z + 3 + 2i$.
Find and simplify the following.

- a $p(z) + q(z)$
- b $p(z) - q(z)$
- c $p(z) + g(z)$
- d $g(z) - q(z)$
- e $q(z) \times g(z)$
- f $p(z) \times q(z)$
- g $p(z) \times g(z)$

2 The function $p(z) = (4 + i)z^3 + 3z^2 + (-1 - 3i)z - 5 + 2i$. Find and simplify the following.

- a $p(z) \div (z - 3)$
- b $p(z) \div (z - 2i)$
- c $p(z) \div (z + 1)$
- d $p(z) \div (z - 2 + i)$
- e $p(z) \div (z + 3 - 4i)$

3 The function $p(z) = -4iz^4 + (5 + 2i)z^3 - 3z + 7$. Find and simplify the following.

- a $p(z) \div (z + 2i)$
- b $p(z) \div (z - 3 - i)$
- c $p(z) \div (z + 2 + 2i)$
- d $p(z) \div (z + 4 - 2i)$
- e $p(z) \div z - 5$

4 CAS Three functions are defined as follows:

$p(z) = (5 - 3i)z^4 + (-3 + 3i)z^3 + (2 - 4i)z^2 + (-7 - i)z - 5 + 3i$
$q(z) = (4 + 2i)z^5 + (7 - 5i)z^3 + (3 + 7i)z^2 + 3 - 5i$ and
$g(z) = (9 + i)z^3 + (-3 + 2i)z + 3 - 5i$
Find and simplify the following.

- a $p(z) + q(z)$
- b $p(z) - q(z)$
- c $p(z) + g(z)$
- d $g(z) - q(z)$
- e $q(z) \times g(z)$
- f $p(z) \times q(z)$
- g $p(z) \times g(z)$

Your teacher will tell you when to use your CAS calculator in the rest of this exercise.

5 Example 11 The function $p(z) = (3 + 5i)z^3 + (-3 - 4i)z^2 + (1 - 5i)z - 8 - 7i$. Find the remainders when $p(z)$ is divided by each of the following.

a $z + 4$ b $z - 2i$ c $z + 3 - 6i$ d $z - 1 - i$ e $z + 3 + 2i$

6 The function $q(z) = (1 - 3i)z^4 + 2z^2 - 5iz + 5 + 4i$. Find the remainders when $q(z)$ is divided by each of the following.

a $z - 3i$ b $z + 1 - 6i$ c $z - 3 - i$ d $z + 2 + 6i$ e $z - 4 + 5i$

7 The function $g(z) = 4z^5 - 2iz^4 + (-2 + 7i)z^3 + 5iz^2 - 6z - 2 + 2i$. Find the remainders when $g(z)$ is divided by each of the following.

a $z + 2i - 5$ b $z - 3i + 1$ c $z - 5 - 3i$ d $z + 6 + 4i$ e $z - 1 - i$

8 Example 12 Show that $z - 2 - 3i$ is a factor of $p(z) = 3iz^3 + (14 - 6i)z^2 + (-12 - 16i)z + 1 + 8i$.

9 Show that $z + 2 - 4i$ is a factor of $p(z) = (3 + i)z^4 + (8 - 13i)z^3 + (-16 + 6i)z^2 + (13 + 6i)z - 14 + 8i$

10 Show that $z + 1 + 5i$ is a factor of
$p(z) = (3 + 2i)z^5 + (-12 + 17i)z^4 + (-5 - 23i)z^3 + (-12 - 3i)z^2 + (23 - 12i)z - 15 + 3i$.

11 Show that $z + 7$ and $z - 2i$ are both factors of
$p(z) = z^4 + (7 - 7i)z^3 + (-7 - 45i)z^2 + (-41 + 22i)z + 56 - 42i$.

12 Show that $z + 2 - 3i$ and $z - 5 + 3i$ are both factors of
$p(z) = (1 + 3i)z^4 + (-3 - 6i)z^3 + (-69 + 9i)z^2 + (-48 - 3i)z + 5 - 105i$.

Reasoning and communication

13 Write $z^2 + 4$ in factored form to show that it is a factor of
$g(z) = (2 + 3i)z^5 + 5iz^4 + (4 + 12i)z^3 + (2 + 21i)z^2 - 16z + 8 + 4i$.

14 Use the quadratic formula to write $z^2 + z + 1$ in factored form to show it is a factor of
$p(z) = (3 + 2i)z^3 + (8 - i)z^2 + (8 - i)z + 5 - 3i$.

15 Write $z^2 + 2z + 4$ in factored form to show it is a factor of
$q(z) = (-3 + i)z^4 + (-2 + 2i)z^3 + (-4 + 2i)z^2 + (16 - 4i)z - 8i$.

6.05 SOLUTION OF POLYNOMIAL EQUATIONS WITH REAL COEFFICIENTS

You already know the **discriminant** of the **quadratic** equation $5x^2 - 2x + 1$, $\Delta = -16$, shows that the solutions are complex. Polynomial equations with real coefficients can have solutions that are real, imaginary or complex.

9780170250306

Example 13

Solve the equation $5z^3 - 12z^2 + 5z - 2 = 0$

Solution

Write the polynomial for the equation.	$p(z) = 5z^3 - 12z^2 + 5z - 2$
Find a factor using the remainder theorem.	$p(1) = -4, p(-1) = -24, p(2) = 0$
Use the factor theorem.	Since $p(2) = 0$, $z - 2$ is a factor.
You can factorise $p(z)$ by inspection.	$p(z) = (z-2)(5z^2 - 2z + 1)$
Write the factored equation.	$(z-2)(5z^2 - 2z + 1) = 0$
Use the null factor law.	$z - 2 = 0$ or $5z^2 - 2z + 1 = 0$
Solve the first equation.	$z - 2 = 0 \Rightarrow z = 2$
Solve the second equation.	$5z^2 - 2z + 1 = 0$
Write the quadratic formula for z.	$\Rightarrow z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute values.	$= \dfrac{2 \pm \sqrt{4 - 4 \times 5 \times 1}}{2 \times 5}$
Simplify.	$= \dfrac{2 \pm \sqrt{-16}}{10}$
Use $i = \sqrt{-1}$	$= \dfrac{2 \pm 4i}{10} = \dfrac{1 \pm 2i}{5}$
Write the answer.	The solutions are 2, $\dfrac{1+2i}{5}$ or $\dfrac{1-2i}{5}$.

Notice that the complex roots in Example **13** are complex conjugates.

The Fundamental theorem of algebra states that any polynomial equation has at least one root (perhaps in the complex number system). It follows immediately that every polynomial of degree n has exactly n roots, some of which may be equal. The proof of the theorem is beyond the scope of this course, but it is of immense importance in mathematics.

You can divide through by the coefficient of the leading term of a polynomial equation of degree n to make any polynomial equation into a monic equation with a leading coefficient of 1. In that case, the equation can be written as $(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n) = 0$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation. For the constant term of the equation to be real, the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ must be real or form conjugate pairs.

IMPORTANT

Every solution of a polynomial equation with real coefficients is either real or has another root that is its complex conjugate. It follows that every real polynomial can be factored to give linear or quadratic factors with real coefficients.

It follows from the above that polynomial equations of odd degree with real coefficients must have at least one real root. The fundamental theorem and its corollaries guarantee the existence of solutions, but it doesn't help in actually finding any solutions. This also applies to factorising real polynomials. However, if you can use the factor theorem to find one real root of a **cubic** power polynomial equation, two of a **quartic** or three of a **quintic** equation, you will be able to find the other roots using the quadratic formula. When you are looking for real roots, they must be factors of the constant term.

Example 14

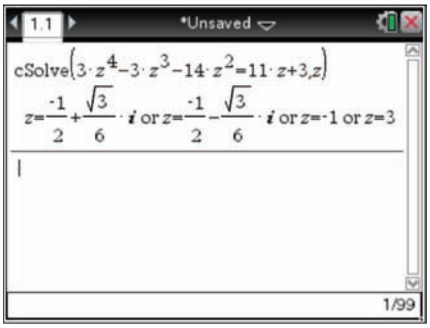
Solve $3z^4 - 3z^3 - 14z^2 = 11z + 3$

Solution

Write the polynomial function.	Let $p(z) = 3z^4 - 3z^3 - 14z^2 - 11z - 3$
Try factors of the constant.	$p(1) = -28, p(-1) = 0, p(3) = 0$
Write the factors.	$(z+1)$ and $(z-3)$ are factors of $p(z)$.
Find the product.	$(z+1)(z-3) = z^2 - 2z - 3$
Write the polynomial as a product.	$p(z) = (z^2 - 2z - 3)(az^2 + bz + c)$
Use the z^4 and constant terms.	$1 \times a = 3$ and $-3 \times c = -3$
Solve for a and c.	$a = 3$ and $c = 1$
Use $p(1)$.	$-28 = (1 - 2 - 3) \times (3 + b + 1)$
Solve for b.	$b = 3$
Write the equation in factored form.	$(z+1)(z-3)(3z^2 + 3z + 1) = 0$
Solve.	$z = -1, z = 3, z = \dfrac{-3 \pm \sqrt{9-12}}{6}$
Write the answers.	$z = -1, z = 3, -\dfrac{1}{2} - \dfrac{\sqrt{3}}{6}i$ or $z = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{6}i$

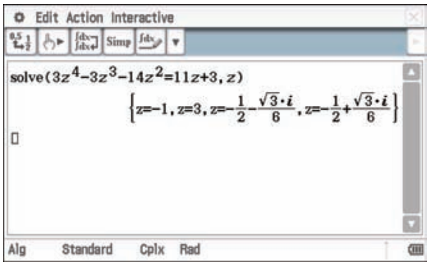
TI-Nspire CAS

Make sure that your calculator is set on complex (rectangular) mode. Use menu, 3: Algebra, C: Complex and 1: Solve or type csolve(and put in the equation and variable for which to solve.



ClassPad

Make sure that your calculator is set to **Cplx** mode. Tap **Action**, **Equation/Inequality** and **solve** or type solve(and enter the equation and variable that you wish to solve for.



Notice in Example **14** that it was not necessary to test $p(-3)$ because $p(3) = 0$ so $z - 3$ was a factor, and so $z + 3$ could not be a factor as well and give the constant -3. Also, the complex roots *are indeed* complex conjugates. You can factorise a cubic, quartic or quintic polynomial in a similar way to solving polynomial equations if you can find one, two or three real factors respectively.

Example 15

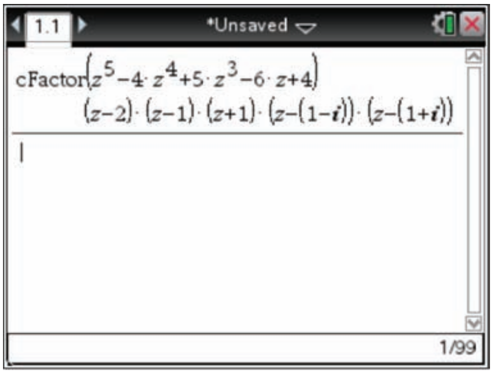
Express $z^5 - 4z^4 + 5z^3 - 6z + 4$ as a product of linear factors.

Solution

Write as a polynomial.	Let $p(z) = z^5 - 4z^4 + 5z^3 - 6z + 4$
Try factors of the constant term, 4.	$p(1) = 0,\ p(-1) = 0,\ p(2) = 0,\ p(-2) = -120$
Write as a product.	$p(z) = (z-1)(z+1)(z-2)(az^2 + bz + c)$
Multiply the linear factors.	$= (z^3 - 2z^2 - z + 2)(az^2 + bz + c)$
Equate z^5 and z^0 terms.	$z^5 = z^3 \times az^2$ and $4 = 2 \times c$
Solve for a and c.	$a = 1$ and $c = 2$
Use $p(-2)$.	$-120 = (-8 - 8 + 2 + 2) \times [4 + b \times (-2) + 2]$
Solve for b.	$b = -2$
Write the quadratic factor.	Quadratic factor $= (z^2 - 2z + 2)$
Use the quadratic formula.	$= \left(z - \dfrac{2+\sqrt{4-8}}{2}\right)\left(z - \dfrac{2-\sqrt{4-8}}{2}\right)$
Simplify.	$= [z - (1+i)][z - (1-i)]$
Write the answer.	$z^5 - 4z^4 + 5z^3 - 6z + 4$ $= (z-1)(z+1)(z-2)(z-1-i)(z-1+i)$

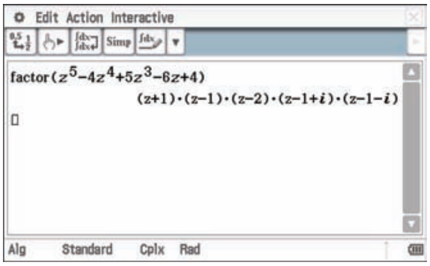
TI-Nspire CAS

In complex (rectangular) mode, use menu, 3: Algebra, C: Complex and 2: Factor or type cfactor(and put in the expression.



ClassPad

Make sure your calculator is set to **Cplx** mode. Tap **Action**, **Transformation**, **factor** and **factor** or type factor(and enter the expression. Press EXE to perform the factorisation.



In Example **15**, since $p(2) = 0$, there was no need to test $p(4)$ or $p(-4)$.

EXERCISE 6.05 Solution of polynomial equations with real coefficients

Concepts and techniques

WS

Polynomials with real coefficients

1 Example 13 For $2z^3 - z^2 - 2z - 8 = (z-2)Q(z)$, what is the quadratic factor?

A $(z^2 + z + 4)$ B $(2z^2 + z + 4)$ C $(2z^2 - z - 4)$
D $(2z^2 - 3z + 4)$ E $(2z^2 + 3z + 4)$

2 Example 14 For $z^4 - 4z^3 - 5z^2 + 20z + 12 = (z+2)(z-3)Q(z)$, what is the quadratic factor?

A $z^2 + 3z - 2$ B $z^2 - 3z - 2$ C $z^2 + 3z + 2$
D $z^2 - 10z - 2$ E $z^2 + 5z - 2$

3 Example 15 For $3z^5 - 13z^4 + 21z^3 - 19z^2 + 12z - 4 = (z-1)(z-1)(z-2)Q(z)$, what is the quadratic factor?

A $z^2 - 3z + 2$ B $3z^2 - z - 2$ C $3z^2 - z + 2$
D $3z^2 - 3z + 2$ E $3z^2 + z + 2$

9780170250306

Reasoning and communication

Your teacher will tell you when to use your CAS calculator for the rest of this exercise.

4 Solve the equation $z^3 - 2z^2 + z - 2 = 0$.

5 Solve the equation $z^3 + z^2 - z + 15 = 0$.

6 Solve the equation $z^4 - 4z^3 + 4z^2 + 4z - 5 = 0$.

7 Solve the equation $z^4 - 7z^3 + 14z^2 + 2z - 20 = 0$.

8 Solve the equation $z^5 - 2z^4 + 40z^2 - 41z - 78 = 0$.

9 Solve the equation $z^5 - 8z^3 - 10z^2 - 9z + 90 = 0$.

10 Express $z^3 + 5z^2 + 8z + 6$ as a product of linear factors.

11 Express $z^4 - 7z^3 + 22z^2 - 32z + 16$ as a product of linear factors.

12 Express $z^5 - z^4 + 4z^3 + 14z^2 - 32z - 40$ as a product of linear factors.

6.06 SOLUTION OF COMPLEX POLYNOMIAL EQUATIONS

A complex polynomial equation can have real solutions. Just because some or all of the coefficients are complex does not mean that the equation cannot have real solutions as well as complex solutions. You should also be on the look out for possible common factors, perfect squares, difference of squares, grouping, trinomials and sums and differences when factorising polynomials and finding solutions of polynomial equations.

Example 16

Solve $z^3 + (2 - 6i)z^2 = (9 + 12i)z + 18$

Solution

Write as a polynomial.

Let $p(z) = z^3 + (2 - 6i)z^2 + (-9 - 12i)z - 18\ (= 0)$

Try factors of -18.

$p(1) = -24 - 18i$ $\quad p(-1) = -6 + 6i$

$p(2) = -20 - 48i$ $\quad p(-2) = 0$

There is a real factor!

$z + 2$ is a factor.

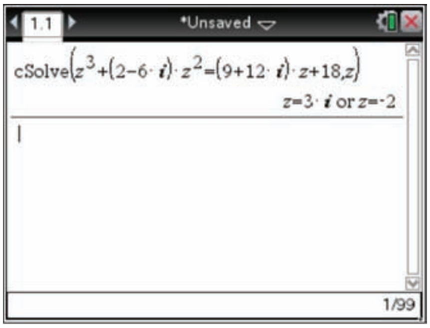
Use long division.

$$
\begin{array}{r l}
 & z^2 \qquad\qquad -6iz \quad -9 \\
z+2 \big) & z^3 + (2-6i)z^2 + (-9-12i)z - 18 + 0i \\
 & \underline{z^3 + (2-0i)z^2} \\
 & \qquad -6iz^2 + (-9-12i)z - 18 + 0i \\
 & \qquad \underline{-6iz^2 + (\ 0-12i)z} \\
 & \qquad\qquad (-9+0i)z - 18 + 0i \\
 & \qquad\qquad \underline{(-9+0i)z - 18 + 0i} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

Write in factored form.	$p(z) = (z + 2)(z^2 - 6iz - 9)$
The quadratic is a perfect square.	$= (z + 2)[z^2 - 2 \times 1 \times 3i + (3i)^2]$
Factorise.	$= (z + 2)(z - 3i)^2$

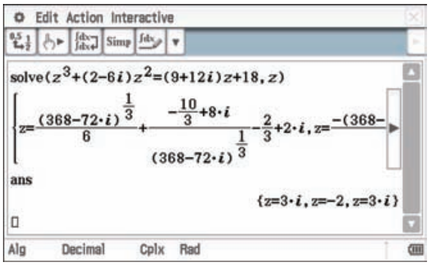
TI-Nspire CAS

Make sure that your calculator is set in complex (rectangular) mode. Use menu, 3: Algebra, C: Complex and 1: Solve or type csolve(and put in the equation and variable.



ClassPad

Make sure that your calculator is set to **Cplx** mode. Tap **Action**, **Equation/Inequality** and **solve** or type solve(and enter the equation and variable. If the result looks complicated, change mode from **Standard** to **Decimal** and output the answer again. In the screen shot opposite, the first expression for the solution was obtained from **Standard** mode, the second from **Decimal** mode.



The ClassPad uses the complex formula to solve cubics, which gives an answer in surd form. By changing to decimal, you can get approximate values that may be easier to compare.

If you can't find a simple real root, look for simple imaginary roots to find factors of complex polynomials.

Example 17

Solve $6iz^4 + 4iz^2 = 7z^3 + 7z + 2i$

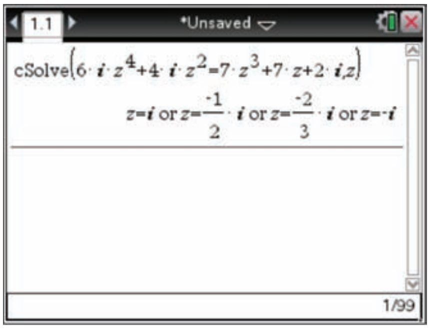
Solution

Write as a polynomial.	Let $p(z) = 6iz^4 - 7z^3 + 4iz^2 - 7z - 2i$
Try real factors of $-2i$.	$p(1) = -14 + 8i$ $\quad p(-1) = 14 + 8i$ $p(2) = -70 + 110i$ $\quad p(-2) = 70 + 110i$
Try imaginary factors of $-2i$.	$p(i) = 0$, $p(-i) = 0$, which is enough to get a quadratic.
Write in factored form.	$p(z) = (z - i)(z + i)(az^2 + bz + c)$
Multiply out the linear ones.	$= (z^2 + 1)(az^2 + bz + c)$

Use z^4 and z^0 terms.	$6iz^4 = z^2 \times az^2$ and $-2i = 1 \times c$
Solve for a and c.	$a = 6i$ and $c = -2i$
Use $p(1)$.	$-14 + 8i = (1+1)(6i + b \times 1 - 2i)$
Simplify.	$2(b + 4i) = -14 + 8i$
Solve for b.	$b = -7$
Write $p(z)$ with all the values.	$p(z) = (z - i)(z + i)(6iz^2 - 7z - 2i)$
Factorise $6iz^2 - 7z - 2i$.	$6i \times (-2i) = 12$ and $-3 \times (-4) = 12, -3 + (-4) = -7$
Write the trinomial in fraction form and cancel.	$6iz^2 - 7z - 2i = \dfrac{(6iz-3)(6iz-4)}{6i}$ $= \dfrac{(2iz-1)(3iz-2)}{i}$
It doesn't matter which factor you multiply by $-i$.	$= \dfrac{-i(2iz-1)(3iz-2)}{i \times (-i)}$ $= (2z + i)(3iz - 2)$ $= (2iz - 1)(-3z + 2i)$
Write $p(z)$ as fully factored.	$p(z) = (z - i)(z + i)\ (2z + i)(3iz - 2) = 0$
Multiply by $-i$ to get $3z$.	$-i(z - i)(z + i)\ (2z + i)(3iz - 2) = 0$
Simplify.	$(z - i)(z + i)\ (2z + i)(3z + 2i) = 0$
Write the answers.	$z = i, z = -i, z = -\frac{1}{2}i$ or $z = -\frac{2}{3}i$

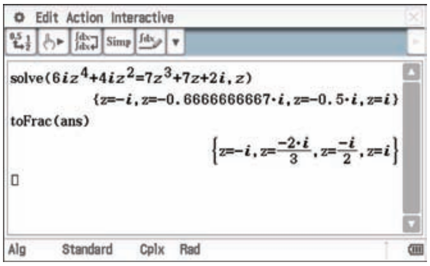
TI-Nspire CAS

Make sure your calculator is set in complex (rectangular) mode. Use menu, 3: Algebra, C: Complex and 1: Solve or type csolve(and put in the equation and variable.



ClassPad

Make sure that your calculator is set to **Cplx** mode. Tap **Action**, **Equation/Inequality** and **solve** or type solve(and enter the equation and variable. Tap **Action**, **Transformation**, **Fraction** and **toFrac** followed by ans) and EXE to convert the decimal expressions to fractions.



CAS calculators use approximate methods for higher power equations, so may give answers as approximations. Change to fractions for obvious cases to get exact results.

EXERCISE 6.06 Solution of complex polynomial equations

WS Real and imaginary factors

Concepts and techniques

Your teacher will tell you how to use your CAS calculator in this exercise.

1 Example 16 The real factors of $iz^4 + 3iz^3 - 3z^2 + (-3 - 4i)z + 6$ are:

A $(z - 1)$ and $(z + 2)$
B $(z + 1)$ and $(z - 2)$
C $(z + 1)$, $(z - 2)$ and $(z - 3)$
D $(z + 1)$, $(z + 2)$ and $(z + 3)$
E $(z - 1)$, $(z - 2)$ and $(z - 3)$

2 Example 17 The imaginary factors of $2z^4 + iz^3 - z^2 - iz - 1$ are:

A $(z + i)$
B $(z - i)$
C $(z + i)$ and $(2z - i)$
D $(z + i)$, $(z - i)$ and $(z + 2i)$
E $(z - i)$, $(z + i)$ and $(z - 2i)$

Reasoning and communication

3 Solve $z^3 - 2iz^2 + (-3 - 2i)z + 2 + 4i = 0$

4 Solve $z^3 + (-3 + 3i)z^2 + (-4 - 3i)z + 12 - 18i = 0$

5 Solve $z^3 + (-2 - 3i)z^2 + (1 + 4i)z - 6 - 3i = 0$

6 Solve $z^3 + (3 + 5i)z^2 + (-8 + 9i)z - 6 - 4i = 0$

7 Solve $z^4 + (-5 - 2i)z^3 + 4iz^2 + (20 + 10i)z - 16 - 12i = 0$

8 Solve $10z^4 + 13iz^3 - 37z^2 - 52iz - 12 = 0$

9 Solve $12z^4 - 41iz^3 + 4z^2 - 69iz - 36 = 0$

10 Solve $z^4 + (-1 - 4i)z^3 + (-1 + 4i)z^2 + (-11 - 42i)z - 72 + 30i = 0$

9780170250306

CHAPTER SUMMARY
COMPLEX ROOTS AND POLYNOMIALS

6

- The nth **roots of unity** are complex numbers z such that $z^n = 1$. They lie on the unit circle and have the form $\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$, where $k = 0, 1, 2, \ldots, n - 1$.
- The principal values of the roots of unity are given by
  $\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$, where $k = 0, 1, 2, \ldots, n - 1$,
  with principal values
  $\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$, where $\pi < \frac{2k\pi}{n} \leq \pi$.
- The polar form of a complex number is also known as the **trigonometric form**.
- The nth **roots of a complex number** z are on a circle of radius $\sqrt[n]{|z|}$, separated from each other by the angle $\frac{2\pi}{n}$ and have the form
  $\sqrt[n]{|z|}\left[\cos\left(\frac{\arg(z)+2k\pi}{n}\right)+i\sin\left(\frac{\arg(z)+2k\pi}{n}\right)\right]$, where $k = 0, 1, 2, \ldots, n - 1$.
- A polynomial is a sum of terms containing non-negative integer powers of the variable. A polynomial in z is of the form $P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z^1 + a_0$, where n is a positive whole number, $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are complex numbers and $a_n \neq 0$.
- The highest power of the variable in a complex polynomial is called the **degree** of the polynomial. The term that contains the highest power is called the **leading term**.
- Complex polynomials may be added, subtracted, multiplied and divided. Division can be set out as long division with allowance for real and complex parts in the coefficients.
- The **division identity** states that if a complex polynomial $P(z)$ is divided by the complex polynomial $D(z)$, then the **remainder** $R(z)$ has degree less than $D(z)$ and the **quotient** $Q(z)$; **divisor**, remainder and **dividend** are related by $P(z) = D(z)Q(z) + R(z)$.
- The **remainder theorem** states that if a complex polynomial $P(z)$ is divided by $D(z) = z - a$, where $a \in \mathbf{C}$, then the remainder is given by $R = P(a)$.
- The **factor theorem** states that $z - a$, where $a \in \mathbf{C}$, is a factor of the complex polynomial $P(z)$ if and only if $P(a) = 0$.
- Every solution of a polynomial equation with real coefficients is either real or has another root that is its conjugate complex. It follows that every real polynomial can be factored to give linear or quadratic factors with real coefficients.
- Cubic, quartic or quintic complex polynomial equations are solvable using quadratic methods for the remaining roots if one, two or three roots respectively can be found.

6 CHAPTER REVIEW
COMPLEX ROOTS AND POLYNOMIALS

Multiple choice

1 Examples 1, 3 Which of the following are 6th roots of 1?

I -1 II $\frac{1}{2}\left(\sqrt{2}+i\sqrt{2}\right)$ III $\frac{1}{2}\left(1-i\sqrt{3}\right)$ IV $\frac{1}{2}\left(\sqrt{3}-i\right)$

A **I** only B **I** and **II** only C **I** and **III** only
D **I** and **IV** only E All except **II**

2 Example 4 Which of the following are 4th roots of -1?

I i II $\frac{1}{2}\left(\sqrt{2}+i\sqrt{2}\right)$ III $\frac{1}{2}\left(-\sqrt{2}-i\sqrt{2}\right)$ IV $-i$

A **I** only B **II** and **III** only C **I** and **III** only
D **I** and **IV** only E All of **I** to **IV**

3 Example 7 The function $p(z) = -2iz^3 + (5 - 2i)z^2 - 3z^4 - 7i$. Which of the following is true?

A $p(z)$ is of degree 4 and has leading term $-2iz^3$.
B $p(z)$ is of degree 4 and has leading term $-7i$.
C $p(z)$ is of degree 3 and has leading term $-2iz^3$.
D $p(z)$ is of degree 4 and has leading term $-3z^4$.
E $p(z)$ is of degree 2 and has leading term $(5 - 2i)z^2$.

4 Examples 8, 9 The function $p(z) = z^4 - 2iz^3 + (2 + i)z^2 + 6 + i$. Which of the following is true?

A $p(1) = 7 + i$ B $p(-1) = 9 - 4i$ C $p(-i) = 7$
D $p(i) = 3i$ E $p(2i) = 2 + 3i$

5 Example 10 Consider the functions $p(z) = (2 - i)z^3 + (5+ 3i)z^2 + (3 + 4i)z - 5 + 3i$ and $q(z) = (-2 + i)z^3 + 5iz^2 - 2z + 1$. Which of the following is true?

A $p(z) + q(z)$ has degree 2 and $p(z)q(z)$ has degree 6
B $p(z) - q(z)$ has degree 3 and $p(z)q(z)$ has degree 3
C $p(z) + q(z)$ has degree 6 and $p(z)q(z)$ has degree 6
D $p(z) + q(z)$ has degree 3 and $p(z) - q(z)$ has degree 3
E $p(z) + q(z)$ has degree 2 and $p(z) - q(z)$ has degree 2

6 Example 11 Consider the function $p(z) = z^4 - 2iz^3 + (2 + i)z^2 + 6 + i$. The remainder when $p(z)$ is divided by $z - 1 + i$ is:

A $4 + 9i$ B $-4 - 9i$ C i
D $8 - 7i$ E $-4 + i$

7 Example 14 For $z^4 - 2z^3 - z^2 + 10z - 8 = (z - 1)(z + 2)Q(z)$, what is the quadratic factor?

A $z^2 - 3z + 4$ B $z^2 + 3z - 4$ C $z^2 - 5z + 4$
D $z^2 - 4z - 4$ E $z^2 + 4z + 5$

8 Example 15 For $3z^5 - 7z^4 + z^3 + 3z^2 - 4z + 4 = (z + 1)(z - 1)(z - 2)Q(z)$, what is the quadratic factor?

A $3z^2 - z - 2$ B $3z^2 - 2z + 3$ C $3z^2 - 2z + 2$
D $3z^2 + 2z + 3$ E $3z^2 - z + 2$

9780170250306

9 Example 16 The real factors of $6iz^4 + (-1 - 6i)z^3 + (1 - 10i)z^2 + (2 - 2i)z - 4i$ are:

A $(z - 1)$ and $(z + 2)$ B $(z + 1)$ and $(z - 2)$ C $(z + 1)$, $(z - 2)$ and $(z - 3)$

D $(z + 1)$, $(z + 2)$ and $(z + 3)$ E $(z - 1)$, $(z - 2)$ and $(z - 3)$

10 Example 17 The imaginary factors of $6z^4 + (-1 + 6i)z^3 + (34 - i)z^2 + (-6 - 2i)z - 12$ are:

A $(z + 2i)$ and $(z - 3i)$ B $(z - 2i)$ only C $(z + 2i)$ only

D $(z + i)$, $(z - 2i)$ and $(z + 3i)$ E $(z - 2i)$ and $(z + 3i)$

Short answer

11 Example 5 a Find the cube roots of -216.

b Show the roots on the complex plane.

12 Example 6 a Solve the equation $z^4 + \frac{9\sqrt{2}}{2} = \frac{9\sqrt{2}}{2}i$.

b Show the solutions on the complex plane.

13 Example 8 Find the values of $g(z) = iz^3 - 2z^2 + 3z + i - 5$ at $z = 1$, $z = -2i$ and $z = 2 + i$.

14 Example 9 CAS For $p(z) = (5 - 2i)z^4 - 3iz^3 + (2 + 3i)z^2 + 2z + 4i$, find $p(2)$, $p(-i)$ and $p(-3 + i)$.

15 Example 10 For $p(z) = (1 - 2i)z^3 + 3iz^2 - 4z + 2i$, $q(z) = -3iz^2 + (4 + i)z + 3$ and $d(z) = z - 1 + i$, find and simplify the following.

a $p(z) + q(z)$ b $p(z)q(z)$ c $p(z) \div d(z)$

16 Example 11 For $p(z) = (-4 - 3i)z^3 + 5z^2 - 4iz - 8$, find the remainders when $p(z)$ is divided by

a $z + 1$ b $z - 3i$ c $z - 2 - i$

17 Example 12 Show that $z + 1 - 2i$ is a factor of $z^3 + (3 - 5i)z^2 + (-4 - 13i)z - 12 - 6i$.

Application

18 Show that no complex numbers have real roots.

19 Show that $z^2 + 2z + 3$ is a factor of $2z^4 + (-2 - 3i)z^3 + (-7 - 3i)z^2 + (-20 - 3i)z - 3 + 9i$.

20 Solve $z^3 = 2z^2 + 9$.

21 Solve $z^4 + (9 - 12i)z + 18 = (1 - 6i)z^3 + (11 + 6i)z^2$.

22 Solve $6z^4 + 30z^2 + 5iz^3 + 20iz + 24 = 0$.

Practice quiz